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LIQUID FILTRATION IN AN UNBOUNDED WATER-BEARING STRATUM WITH AN INCLINED CONFINING BED

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Polubarinova-Kochina, Numerov, and other authors paid much attention to filtration problems of a heavy incompressible liquid in inclined water-bearing strata. In this work, therefore, classical schemes of liquid filtration on inclined confining beds are considered along with the general problem of filtration for arbitrary polygonal impermeable walls of a water-bearing stratum. In doing so, we also consider direct problems of physical and geometrical parameters of filtration flows.

Key words: *liquid filtration, analytical function, a priori estimates, solvability, local uniqueness.*

1. GENERAL PROBLEM OF FILTRATION

We start from the general problem of liquid filtration in an unbounded porous layer studied in [1] assuming that the angles $-\alpha_s\pi$ and $-\alpha_m\pi$ at the apices $z_s = \infty$ and $z_m = \infty$ of the polygon P formed by infinite segments P_k^*, P_{k+1}^* ($k = s,\, m$) are arbitrary, i.e., $\alpha_s \neq 0$ and $\alpha_m \neq 0$ (Fig. 1).

The filtration domain D is bounded by a free (unknown) boundary L and by a specified polygon P formed by permeable walls of a water-bearing stratum (P^1 and P^3), an impermeable top of the water-bearing stratum adjoining to them ($P_0^1 \cup P_0^3$), $P_0^k \cap P^k = z_0^k$ ($k = 1, 3$), and an impermeable foot of the water-bearing stratum P^2.

We direct the x axis opposite to the vector of gravity acceleration and assume that $z = x + iy$. Let us denote the apices of the polygon P by z_k ($k = \overline{0, n+1}$), the angles at them by $\alpha_k\pi$, and the lengths of finite segments of the polygon P by $l_k = |z_k - z_{k-1}|$. The points z_0^k ($k = 1,\, 3$) are also apices of the polygon P: $z_0^1 = z_{s_0}$ ($0 < s_0 < s$) and $z_0^3 = z_{m_0}$ ($m < m_0 < m$).

Let $z_s = P_0^1 \cap P^2$ and $z_m = P^2 \cap P_0^3$ ($0 < s < m < n+1$) be points at infinity upstream and downstream, respectively ($z_m = z_s = \infty$).

For each of the infinite segments P_s and P_{s+1} ending at the points (z_{s-1}, z_s), (z_s, z_{s+1}) and segments P_m and P_{m+1} ending at the points (z_{m-1}, z_m), (z_m, z_{m+1}), we fix two finite points (z_{s-1}^*, z_s^*), (z_{s+1}^*, z_{s+2}^*) and (z_{m-1}^*, z_m^*), (z_{m+1}^*, z_{m+2}^*) and include them into the number of apices of P with angles at them equal to π.

An analytical function $w(z) = \varphi + i\psi$ ($z = x + iy$) is sought in the domain D as the complex potential of filtration.

In the plane of the complex potential $w = \varphi + i\psi$, to the filtration domain D there corresponds a benched band D^* bounded by the straight lines $\psi = \text{const}$ and $\varphi = \text{const}$: $\{-\infty < \varphi < \infty,\ \psi = 0\} \equiv w(P^2)$ (the image of the confining bed P^2); $\{\varphi < \varphi_{s_0},\ \psi = Q_s\} \equiv w(P_0^1)$, $\{\varphi = \varphi_0,\ Q_s^- < \psi < Q_s^+\} \equiv w(P^1)$, $\{\varphi_0 < \varphi < \varphi_{n+1},\ \psi = Q_0\} \equiv w(L)$, $\{\varphi = \varphi_{n+1},\ Q_m^- < \psi < Q_m^+\} \equiv w(P^3)$, and $\{\varphi_{n+1} < \varphi,\ \psi = Q_m\} \equiv w(P_0^3)$. Here $\varphi_k = \operatorname{Re} w_k$ are the given values of bottom drive, $Q_j^- = \min(Q_0, Q_j)$ and $Q_j^+ = \max(Q_0, Q_j)$ ($j = s, m$), Q_0, Q_s, and Q_m are the sought values of water discharge under a dam (Q_0) and at the upper (Q_s) and lower (Q_m) infinite apices z_s and z_m of the porous layer. The given scheme of the filtration flow determines the values of the angles $\gamma_j\pi$ at the finite apices $w_j \in \partial D^*$ ($j = 0,\, n+1,\, s-1,\, m+1$) equal to $\pi/2$ or $3\pi/2$. Here, the conditions

$$\gamma_0 + \gamma_{s-1} = \gamma_{n+1} + \gamma_{m+1} = 2$$

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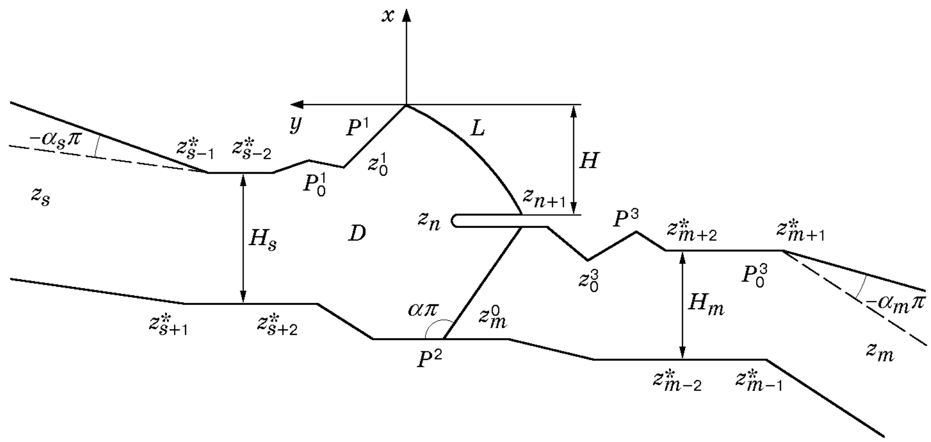

Fig. 1

must be satisfied, which, in particular, determine the relation between the sought discharges Q_0, Q_s, and Q_m. For example, we have $Q_0 > Q_s$ for $\gamma_0 = 1/2$ and $Q_0 < Q_s$ for $\gamma_0 = 3/2$. Other types of domains D^* are considered in the following sections.

In accordance with the geometry of the domains D and D^*, the representations for derivatives of conformal mappings z: $E \to D$ and w: $E \to D^*$ (E: $\operatorname{Im}\zeta > 0$) take the form [1, 2]

$$\frac{dz}{d\zeta} = \frac{\Pi(\zeta)}{\pi i} \int\limits_{|t|>1} \frac{|\Pi_0(t)|\,dt}{\Pi(t)(t-\zeta)}, \quad \Pi(\zeta) = \prod_{k=0}^{n+1} (\zeta - t_k)^{\beta_k}, \quad \frac{dw}{d\zeta} = \prod_{j=0}^{5} (\zeta - \tau_j)^{\gamma_j - 1}. \tag{1}$$

Here $\beta_k = \alpha_k - 1$ and $\alpha_k\pi$ are the interior angles at the apices and ends z_k ($k = \overline{0, n+1}$) of the polygon P ($z_s = z_m = \infty$, $\alpha_s \leqslant 0$, $\alpha_m \leqslant 0$, and $0 \leqslant s < m \leqslant n+1$), t_k are the preimages of $z_k = z(t_k)$, τ_j are the preimages of the apices $w_j = w(\tau_j)$ of the polygon ∂D^*, coinciding with part of t_k, $\gamma_j = 1/2$ and $3/2$ ($j \neq s$, m) and $\gamma_s = \gamma_m = 0$. Let us fix the constants $t_0 = -1$ and $t_{n+1} = 1$.

2. SYSTEM OF EQUATIONS FOR PARAMETERS

If the vector $T = (t_1, t_2, \ldots, t_n) \in R^n$ of unknown constants t_k ($k = \overline{1, n}$) in representation (1) is arbitrarily fixed, then the corresponding mapping $z = z(\zeta, T)$, z: $E \to D(T)$ transfers the segment $[-1, 1]$ to the polygon $P(T)$ with the sides parallel to the sides of the given polygon P. Let us construct a system of equations relative to the vector T, whose solution secures the equality $P(T) = P$.

We assume that $z_0 = 0$ and specify the lengths $l_k = |z_k - z_{k-1}|$ of the finite segments of the polygon P:

$$l_k = \int\limits_{t_{k-1}}^{t_k} |\Pi(t)|\,|M(t)|\,dt \qquad (k = \overline{1, n},\ k \neq s, m). \tag{2}$$

Since the point $z_0 = 0$ is fixed on the polygon $P^1 \cup P^1_0$, the corresponding conditions (2) completely specify the position of this point. Similarly, to fix the position of the polygons P^2 and $P^3 \cup P^3_0$, let us specify in them the coordinates of the points z_{s+1} and z_{m+1}. Since the boundary condition on the free boundary preimage yields

$$|x(t_{n+1}) - x(t_0)| = |\varphi_{n+1} - \varphi_0| = H \quad (|\ln H| \leqslant N < \infty),$$

the value of $|x(t_{n+1})| = |\operatorname{Re} z_{n+1}| = H$ is given. Therefore, to determine the position of the polygons P^2 and $P^3 \cup P^3_0$, it is sufficient that the following equations should be satisfied:

$$l_s + i l_{s+1} = \int\limits_{t_{s-1}}^{t_{s+1}} \frac{dz}{d\zeta}\,d\zeta, \qquad l_m = \operatorname{Im} \int\limits_{t_0}^{t_{n+1}} \frac{dz}{d\zeta}\,d\zeta. \tag{3}$$

We assume that $u_k = t_k - t_{k-1}$ ($k = \overline{1, n+1}$) and introduce the vector $u = (u_1, \ldots, u_n) \in R^n$, in terms of which the vector $T = (t_1, \ldots, t_n) \in R^n$ is uniquely determined. Then, the vector $u \in R^n$ is a solution of the functional equation

$$l = g(u, \alpha), \tag{4}$$

where l_k and $l = (l_1, \ldots, l_n)$ are represented as (2) or (3); $\alpha = (\alpha_0, \ldots, \alpha_{n+1})$. According to formulas (4) to each fixed vector $u = (u_1, \ldots, u_n)$, $u_k \neq 0$ there corresponds a polygon $P(u)$ coinciding with P only when (4) with the given l_k corresponding to P is satisfied. In this case, the polygons $P(u)$, generally speaking, are multivalent, and some of their segments can have external self-intersection. To take this possibility into account, we introduce some definitions. Let us put infinite segments $P_0 = \{z\colon \operatorname{Re} z = 0, \operatorname{Im} z < 0\}$ and $P_{n+2} = \{z\colon |\operatorname{Re} z| = H, \operatorname{Im} z < \operatorname{Im} z_{n+1}\}$ from the points z_0 and z_{n+1}, replacing the unknown curve L, and construct a polygon $\bar{P} = P \cup P_0 \cup P_{n+2}$.

Let us call the vector $p = (l, \alpha) \in R^{2n+1}$ the geometric characteristics of the polygon P (polygon $\bar{P}$) and impose the following conditions on (l, α):

$$\begin{gathered} 0 < \delta \leqslant \alpha_k \leqslant 2, \quad k \neq s, m, \quad |\ln l_k| \leqslant \delta^{-1} \quad (k = \overline{1, n}), \\ -1/2 + \delta \leqslant (\alpha_s, \alpha_m) \leqslant 0, \quad 1/2 \leqslant \alpha_0 \leqslant 3/2 - \delta, \quad \alpha_{n+1} = 1. \end{gathered} \tag{5}$$

Condition (5) imposed on the slope angle $\alpha_0 \pi$ at the point $z_0 = 0$ and the equality $\alpha_{n+1} = 1$ maintain the boundedness of $M(\zeta)$ in (1) for $\operatorname{Im} \zeta > 0$.

Let the following inequalities be satisfied for every curve $P_{ij} \subset D_0$, $\partial D_0 = \bar{P}$ with the ends on non-adjoining segments $(P_i, P_j) \in \bar{P}$:

$$|P_{ij}| \geqslant \delta > 0, \qquad |i - j| \geqslant 2. \tag{6}$$

If the domain D_0 is nondegenerate, then the derivative of the conformal mapping $Z\colon E \to D_0$ is to be represented in the form

$$\frac{dZ}{d\zeta} = C\Pi(\zeta), \quad C = \text{const}, \quad Z\colon E \to D_0. \tag{7}$$

Polygons P (polygons $\bar{P}$), for which assumptions (5)–(7) are fulfilled will later be called simple, and the class of simple polygons will be denoted by $G(\delta)$ [$P \subset G(\delta)$ or $p = (l, \alpha) \in G(\delta)$].

The simple polygons $P \subset G(\delta)$ and the polygons $\bar{P}$ corresponding to them, generally speaking, are multivalent and admit external self-intersections of nonadjoining segments, i.e., P and $\bar{P}$ may lie on the Riemann surfaces of the zeroth kind.

3. *A PRIORI* ESTIMATES AND LOCAL UNIQUENESS OF THE SOLUTION

Theorem 1 (on *a priori* estimates). *Let Eq.* (4) *for* $(l, \alpha) \in G(\delta)$ *have a solution* $u = (u_1, \ldots, u_n) \in R^n$, $u_k > 0$. *Then, the following inclusion is fulfilled* (*a priori estimates*):

$$u \in \Omega = \{u\colon 0 < \varepsilon(\delta) \leqslant u_k,\ k = \overline{1, n}\}. \tag{8}$$

Proof. We start to prove estimates (8) from verifying their correctness for the parameters τ_k of the conformal mapping $w : E \to D^*$:

$$|\tau_{j+1} - \tau_j| \geqslant \varepsilon > 0, \quad j = \overline{0, 5}\ (\tau_6 = \tau_0). \tag{9}$$

Inequalities (9) ensure nondegeneracy of the domain D^* and are proved by the method of the extremal lengths of the family of curves [3].

Let us consider the filtration domain D (Fig. 1) as a "tetragon" with the apices $z_0, z_{n+1}, z_0^1, z_0^3$, and z_{m+1}^* and map conformally the domain D^* onto the rectangle Ω in the auxiliary plane W ($W\colon D^* \to \Omega$) with the apices $W_k = W(w_k)$, $w_k = \{w_0, w_{n+1}, w_0^1, w_0^3\}$.

First of all, note that the module of the "tetragon" D ($\operatorname{mod} D$) with a pair of the opposite "sides" L and $\Gamma = (P^2 \cup P_0^1 \cup P_0^3)$ is equal to the module of the rectangle Ω by virtue of the conformality of the mapping $W = W(z)$, $W\colon D^* \to \Omega$. To estimate $\operatorname{mod} D$, it is sufficient to determine the extremal length $\lambda(\omega)$ of the family of curves $\{\omega\}$ connecting L and Γ. Along with $\{\omega\}$, we consider the family of curves $\{\omega^*\}$ connecting the "sides"

$P_0 \cup P_{n+2}$ and Γ of the polygon $\bar{P}$, $D \subset D(\bar{P})$ (P_0 and P_{n+2} are the rays emanating from the points z_0 and z_{n+1}). Let us draw a segment of the straight line $x = x_{n+1}$ from the point z_{n+1} until it intersects $P^1 \cup \Gamma$ at the point $z_* \in (P^1 \cup \Gamma)$ (if there is no intersection, $z_* = \infty$).

We assume that $L_* = \{x = x_{n+1},\, y_{n+1} < y < \bar{y}\}$ and $P_*^1 = \{x = x_{n+1},\, \bar{y} < y < y_*\}$, $\bar{y} = (y_* - y_{n+1})/2$. The domain obtained is denoted by $D_* \subset D$.

If $z_* \in P^1$, then we take $\Gamma_* = \Gamma$ as the side opposite to L_*. For $z_* \in \Gamma$, we have $\Gamma_* = \Gamma \setminus l_*$, where $l_* \subset P_0^1$ is part of the segment P_0^1 from the point z_* to the point z_0^1.

Consider the family of curves $\{\omega_*\}$ connecting Γ_* and L_* in the domain $D_* \subset D$. The modules $\lambda(\omega)$, $\lambda(\omega^*)$, and $\lambda(\omega_*)$ of these families are related as follows:

$$0 < \lambda(\omega_*) \leqslant \lambda(\omega) = \operatorname{mod} D = \operatorname{mod} D^* < \lambda(\omega^*) < \infty.$$

By virtue of the fact that the "tetragons" D_* and $D(\bar{P})$ are fixed, the values of $\lambda(\omega_*)$ and $\lambda(\omega^*)$ can be obtained explicitly. The estimates obtained for $\lambda(\omega)$ imply the nondegeneracy of the polygon ∂D^*, which proves the correctness of estimates (9).

To prove estimates (8) for the remaining $u_k = t_k - t_{k-1}$ ($k = \overline{1,n}$), let us assume the opposite, namely, that part of u_k may tend to zero, i.e., the constants t_k corresponding to them converge: $|t_{k+1} - t_k| \to 0$. Let, for the time being, the converging parameters t_k not include $t_0 = -1$ and $t_{n+1} = 1$, i.e., t_k, $1 \leqslant i \leqslant k \leqslant j \leqslant n$ converge. By virtue of (9), the constants $t_s = \tau_2$ and $t_m = \tau_3$ cannot simultaneously be among the converging parameters. Therefore, let us consider the case $z_j \neq \infty$ (while the possibility of $z_i = \infty$ is not excluded), representing $l_j = |z_j - z_{j-1}|$ in the form

$$l_j = \int\limits_{t_j}^{t_{j+1}} \prod_{k=i}^{j} |t - t_k|^{\beta_k}\, |M_j(t)|\, dt \quad (\beta_k = \alpha_k - 1,\ k = \overline{0, n+1}).$$

Here $|M_j(t)| = |\Pi_j(t)|\, |M(t)| \neq 0, \infty$ at $t \in [t_j, t_{j+1})$ and $|\Pi_j(t)| = \prod\limits_{k=0}^{i-1} |t - t_k|^{\beta_k} \prod\limits_{k=j+1}^{n+1} |t - t_k|^{\beta_k}$.

Let us assign all $\beta_k < 0$ to Σ' and all $\beta_k \geqslant 0$ ($k = \overline{i,j}$) to Σ'' and assume that $\nu = -\Sigma'\beta_k$ $\mu = \Sigma''\beta_k$.

Assumptions: $\mu - \nu + 1 \leqslant 0$ and $(t_j - t_i) \to 0$ $(1 \leqslant i < j \leqslant n)$.

By the condition, t_{j+1} and t_j do not converge, consequently, there exists $\varepsilon > 0$ such that $\varepsilon + t_j < t_{j+1}$. Since $|M_j(t)| \geqslant a \neq 0$, $t \in [t_j, t_j + \varepsilon)$; $t - t_j \leqslant t - t_k \leqslant t - t_i$, then,

$$l_j = \int\limits_{t_j}^{t_{j+1}} \Big|\frac{dz}{dt}\Big| dt \geqslant \int\limits_{t_j}^{t_j+\varepsilon} \Big|\frac{dz}{dt}\Big| dt \geqslant a \int\limits_{t_j}^{t_j+\varepsilon} (t - t_i)^{-\nu} (t - t_j)^{\mu}\, dt.$$

We substitute the variables in the last integral assuming that $t = rs + t_j$. Then, we obtain

$$l_j \geqslant a r^{\mu - \nu + 1} \int\limits_0^{\varepsilon/r} s^{\mu} (1 + s)^{-\nu}\, ds \to \infty \quad \text{at} \quad r = (t_j - t_i) \to 0,$$

which contradicts conditions (5).

If $\mu - \nu + 1 = 0$, we obtain in a similar way

$$l_j \geqslant a \int\limits_0^{\varepsilon/r} s^{\mu - \nu} \Big(1 + \frac{1}{s}\Big)^{-\nu} ds \geqslant 2^{-\nu} a \int\limits_1^{\varepsilon/r} \frac{ds}{s} \to \infty \quad \text{at} \quad r \to 0.$$

Assumptions: $\mu - \nu + 1 > 0$ and $(t_j - t_i) \to 0$ $(1 \leqslant i < j \leqslant n)$.

Let us construct a semicircle K_r of radius $r = t_j - t_i$ centered at the point $\zeta_0 = (t_j + t_i)/2$ in the upper half-plane $\operatorname{Im}\zeta > 0$, choosing r sufficiently small for the inequalities $t_i - r/2 > t_{i+1}$ and $t_j + r/2 < t_{j+1}$ to be fulfilled. We have

$$|M_j(\zeta)| \leqslant A < \infty, \quad \zeta \in K_r; \qquad r/2 \leqslant |\zeta - t_k| \leqslant 2r, \quad \zeta \in K_r, \quad k = \overline{i,j}.$$

Consider the curve $\Lambda_r = F(K_r) \subset D$ ($F\colon E \to D$) with the ends on the segments P_{i-1} and P_{j+1} of the polygon P. For $r \to 0$, the length $|\Lambda_r|$ tends to zero, and thus also $l_k \to 0$ ($k = \overline{i,j}$):

$$|\Lambda_r| = \Big|\int\limits_{K_r} \frac{dz}{d\zeta}\, d\zeta\Big| \leqslant \pi r A |\Pi'(\zeta)|\,|\Pi''(\zeta)| \leqslant A\pi r(2r)^{\mu}\Big(\frac{r}{2}\Big)^{-\nu} \to 0.$$

Here only the powers $\beta_k < 0$ enter into $\Pi' = \Pi'(\zeta - t_k)^{\beta_k}$ and all the $\beta_k > 0$ ($k = \overline{i,j}$) into Π''. Thus, the assumption that $r = (t_j - t_i) \to 0$ is incorrect, i.e., there can be found such $\varepsilon > 0$ for which $(t_j - t_i) \geqslant \varepsilon > 0$.

Similarly, we consider the case $z_j = \infty$ and $z_i \neq \infty$ with the side $l_i = |z_i - z_{i-1}|$ taken for $\mu - \nu + 1 \leqslant 0$.

Therefore, we have simply to consider the case where only one of the parameters t_0 or t_{n+1} is among the converging t_k, for example, $t_0 = -1$, i.e., $(t_j - t_0) \to 0$. Note that $0 < j < m$, since according to (9), $t_m - t_s \geqslant \varepsilon > 0$. We assume that

$$M_0(\zeta) = (\zeta - t_0)^{\alpha_0 - 1/2 + \gamma}\Pi_*(\zeta)M(\zeta), \qquad \Pi_0(\zeta) = \prod_{k=1}^{j}(\zeta - t_k)^{\beta_k}(\zeta - t_0)^{\bar\beta_0},$$

where $\Pi_*(\zeta) = \Pi(\zeta)\Pi_0^{-1}(\zeta)$, $\gamma_0 = 0$ for $\alpha_0 > 1/2$ and $0 < \gamma_0 \ll 1$ for $\alpha_0 = 1/2$, and $\bar\beta_0 = -1/2 - \gamma_0$ (below, the bar over β_0 is omitted).

Let us assign all $\beta_k < 0$ to Σ' and all $\beta_k \geqslant 0$ ($k = \overline{0,j}$) to Σ'' - ($\beta_0 = \bar\beta_0$) and denote $\nu = -\Sigma'\beta_k$ and $\mu = \Sigma''\beta_k$.

Assumptions: $\mu - \nu + 1 \leqslant 0$ and $(t_j - t_0) \to 0$ ($1 \leqslant j \leqslant m-1$).

According to the choice of $\bar\beta_0 = -1/2 - \gamma_0$, we have $M(t_0) \neq 0, \infty$; therefore, the inclusion of t_0 into the number of converging parameters does not complicate the proof in the case considered.

Assumptions: $\mu - \nu + 1 > 0$ and $(t_j - t_0) \to 0$ ($1 \leqslant j \leqslant m-1$).

Similarly to the case $r = (t_j - t_i) \to 0$, $t \geqslant 1$, it is found that $|\Lambda_r| = |F(K_r)| \to 0$ as $r \to 0$, where $K_r = \{\zeta\colon \operatorname{Im}\zeta > 0,\ |\zeta - t_0 - r/2| = r\}$, and $F\colon E \to D$.

Since the image $z_*(r)$ of the point $t_0 - r/2 = t_*(r) \in K_r$ lies at the free boundary L, it must be proved additionally that $z_* = F[t_*(r)] \to 0$ as $r \to 0$ [$F(t_0) = 0$]. Taking into account that $\sum\limits_{k=0}^{j}\beta_k = \mu - \nu > -1$, we obtain

$$|F(t_*)| = \Big|\int\limits_{t_*}^{t_0}\Pi_0(t)M_0(t)\,dt\Big| \leqslant \max|M_0| \int\limits_{t_*}^{t_0}\prod_{k=0}^{j}|t - t_k|^{\beta_k}\,dt \to 0$$

as $r \to 0$. Thus, $l_k \to 0$ as $r \to 0$ ($k = \overline{1,j}$).

It is conclusively established that, if $r = (t_j - t_i) \to 0$, then $|\ln l_k| \to \infty$, i.e., there arises a contradiction with condition (5) of the simple polygon P. Theorem 1 is proved.

Theorem 2 (of local uniqueness)**.** *If the solution of Eq.* (4) *exists, then* $g(u,\alpha) \in C^2[\Omega \times G]$ *and this solution is locally unique, i.e.,*

$$\frac{Dg(u,\alpha)}{Du} = \{g_{ij}\} \neq 0, \infty, \qquad g_{ij} = \frac{\partial g_i}{\partial u_j}, \qquad (u,\alpha) \in (\Omega \times G). \tag{10}$$

Proof. Differentiability of $l_k = g_k(u,\alpha)$ with respect to arguments being represented in the form (2) is established in [4], and it is readily checked directly for representation (3).

Relation (10) is proved following the procedure suggested in [4]. We calculate the variation $\delta l = \delta g(u,\alpha)$ of the vector $l \in R^n$ by means of the variation δu of the sought solution $u \in R^n$ for fixed $\alpha \in G(\delta)$: $\delta l = (Dg/Du)\delta u$. Let $\delta u \neq 0$; we calculate δz for $\delta l = 0$:

$$\delta z = \int\limits_{-1}^{\zeta}\Pi(\zeta)\Omega(\zeta,\delta u)\,d\zeta, \quad \Omega = \sum_k\Big[(1-\alpha_k)(\zeta - t_k)^{-1}M(\zeta) + \frac{\partial M}{\partial t_k}\Big]\delta t_k.$$

It is readily verified that $d\delta z/d\zeta$ satisfies the boundary-value problem

$$\arg\frac{d\delta z}{dt} = \gamma_k\pi, \quad t \in [t_k, t_{k+1}]; \qquad \arg\frac{d\delta z}{dt} = 0, \quad |t| \geqslant 1.$$

Consequently, we have

$$\delta z = \prod_{k=0}^{n+1}(\zeta - t_k)^{\alpha_k - \varepsilon_k}Q_p(\zeta), \qquad Q_p = \sum_{k=0}^{p}c_k\zeta^k$$

($\varepsilon_k = 0$ for $\delta t_k = 0$ and $\varepsilon_k = 1$ at $\delta t \neq 0$). On comparing the obtained solution δz of the boundary-value problem with δz calculated above in the vicinity of $\zeta = \infty$, we obtain $Q_p \equiv 0$ and, therefore, $\delta z = 0$. Then, from the representation for δz, we find $\Omega(\zeta) \equiv 0$, hence $\delta u_k = 0$, from which there follow relations (10). Theorem 2 is proved.

4. INITIAL POLYGON

Consider the polygon $P_* = \bigcup_{k=1}^{3} P_*^k$: $P_*^1 = \{z\colon x = 0, y > 0\}$, $P_*^3 = \{z\colon x = -H, y < y_n\}$, $P_*^2 = \{z\colon z = -H_1, -\infty < y < \infty\}$, $H_1 > H$. At the point z_k $(k = 0, n+1)$, the angles are $\alpha_k\pi = \pi$, and at the point z_n, the angle is $\alpha_n\pi = 2\pi$. Then, in (1), we have

$$\Pi(\zeta) = [(\zeta - \tau_2)(\zeta - \tau_3)]^{-1}(\zeta - t_n), \qquad |\Pi_0| = \prod_{k=1}^{4} |t - \tau_k|^{-1/2}.$$

Let us fix $\tau_1 = -1$, $\tau_2 = 0$, and $\tau_4 = 1$ and seek for τ_3 from the equation

$$H = \int_0^{\tau_3} [(1 - t^2)(\tau_3 - t)t]^{-1/2}\, dt.$$

In the integral, we substitute the variables $t = \tau_3(1 - \sigma)$:

$$H = \int_0^1 [1 - \tau_3^2(1 - \sigma)^2]^{-1/2}[\sigma(1 - \sigma)]^{-1/2}\, d\sigma \equiv U(\tau_3).$$

By construction, we have $dU/d\tau_3 > 0$, $U(0) = \int_0^1 [\sigma(1 - \sigma)]^{-1/2}\, d\sigma \equiv H_0$, and $U(1) = \infty$. Let us fix $H > H_0$. Then, from the equation $H = U(\tau_3)$, the constant τ_3 $(\tau_2 = 0 < \tau_3 < 1 = \tau_4)$ is uniquely defined.

Let us now present the specified quantity $H_2 = H_1 - H > 0$ in the form

$$H_2 = \pi\Big|\frac{dz}{d\zeta}(\zeta - \tau_3)\Big|_{\zeta=\tau_3}, \quad H_2 = \pi\tau_3^{-1}(t_n - \tau_3)|M(\tau_3)| \equiv X(t_n) \quad [M(\tau_3) = \varphi(t_n)].$$

We have $dX/dt_n > 0$, $X(\tau_3) = 0$, and $X(\tau_4) = \infty$. Therefore, the equation $H_2 = X(t_n)$ is uniquely solvable with respect to t_n and $\tau_3 < t_n < \tau_4$. Thus, the conformal mapping $z = F_*(\zeta)$, $F_*\colon E \to D_0$, $\partial D_0 = P_0 \cup L_*$ is uniquely defined.

Let us arbitrarily fix the points t_k, $t_0 = -1 < t_1 < \ldots < t_s = \tau_2 < \ldots < t_m = \tau_3 < \ldots < t_{n+1} = 1$ and find their images $z_k = F_*(t_k)$, $k = \overline{0, n+1}$. Based on the points z_k, we construct system (2), (3) uniquely solvable by construction ($\alpha_k = 0$, $k = \overline{1, n-1}$).

5. UNIQUE SOLVABILITY OF THE EQUATION $l = g(u, \alpha)$

Theorem 3 (of existence and uniqueness)**.** *Equation* (4) *corresponding to the simple polygon $P \subset G(\delta)$ and, consequently, the original problem of the filtration theory are uniquely solvable.*

Proof. Proof of the theorem, by virtue of Theorems 1 and 2, follows from the convergence of the continuity method [4, p. 122]. To apply this method, we connect the apices of the initial polygon P_* constructed in Sec. 4 with the respective apices of the original polygon P by smooth nonintersecting curves S_k $(k = \overline{0, n+1})$. By arbitrarily choosing points $z_k(S_k)$ on these curves and connecting them by straight-line segments, we obtain a family of polygons $\{P(S)\}$ $[S = (S_0, \ldots, S_{n+1})]$ with interior angles $\alpha_k\pi$ and lengths of the sides l_k $(k = \overline{0, n+1})$ [l_k for $k = s$, $s+1$, m are calculated by formulas (3)]. By construction, $P(S) \in G(\delta)$. Further, the continuity method consists in the successive proof of unique solvability of Eq. (4) using the theorem of implicit functions for the polygons $P(S)$ starting from P_* continuously deformable along S. By the theorem of uniqueness for the initial polygon, it also holds true for all the polygons $P(S)$ including the initial polygon P [4, p. 122–123]. Theorem 3 is proved.

Further, several particular problems are studied.

6. FILTRATION FLOW OF UNDERGROUND WATER ALONG AN INCLINED CONFINING STRATUM UNDER A HORIZONTAL DRAIN

The filtration scheme is shown in Fig. 2 borrowed from [5, p. 231]. The upstream and downstream depths and discharges of the flow are equal to h, Q, and h_1, Q_1 respectively. The domains D^* of the complex potential

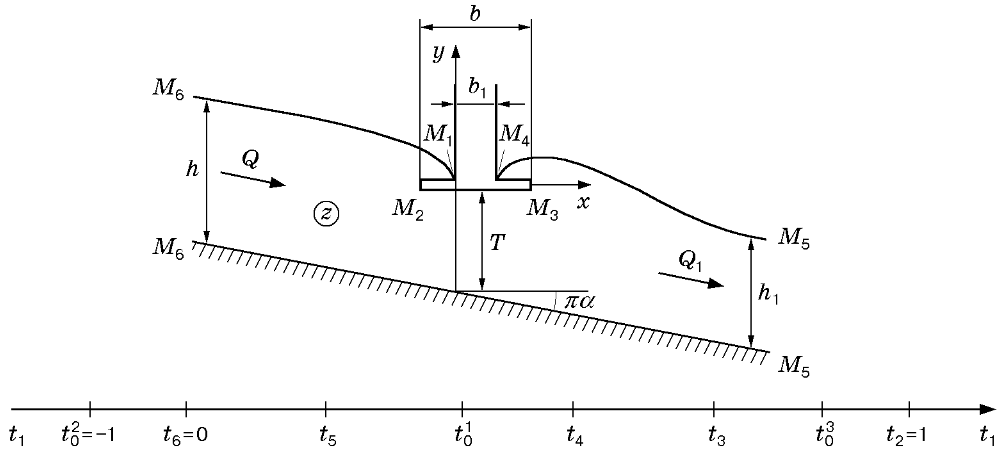

Fig. 2

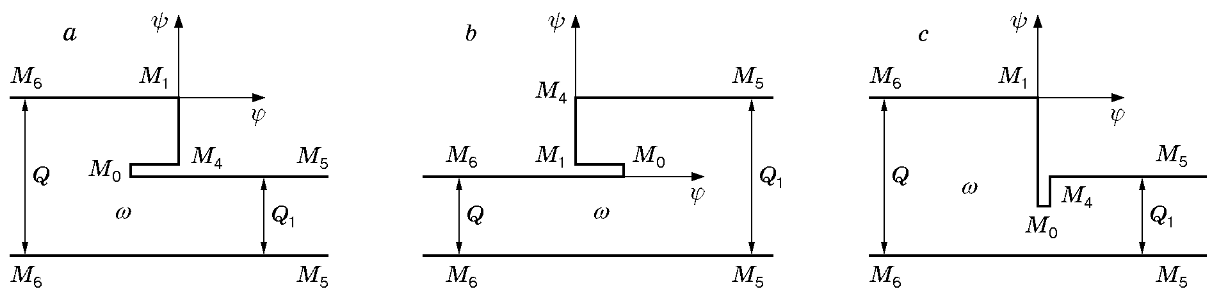

Fig. 3

$\omega = \varphi + i\psi$ are shown in Fig. 3, and the correspondence of the boundary points of conformal mappings Z: $E \to D$ and ω: $E \to D^*$ is shown in Fig. 2 (t_j are the preimages of the points M_j, $j = \overline{0.6}$).

We retain here the notation of variables related to the variables z and w by the formulas $Z = -iz$ and $\omega = kw$ (k is the permeability coefficient), which were used in the monograph [5, p. 231–239].

Depending on the position of the flow branching point M_0, where the flow rate is zero, let us consider three flow schemes described in [5, p. 233–239].

1. Volumetric liquid inflow to the drain (discharge $Q > Q_1$, branching point M_0 is on the right-hand branch of the free boundary M_5M_4) (see Figs. 2 and 3a).

2. Filtration liquid outflow from the drain (channel) into the ground ($M_0 \in M_6M_1$ and $Q < Q_1$) (see Figs. 2 and 3b).

3. Inflow of the groundwater in the upper part of the drain; in the lower part of the drain, the liquid leaks (seeps) from the drain into the ground (M_0 lies on the drain M_3M_2 and $Q > Q_1$) (see Figs. 2 and 3c).

It is assumed that there are points M_2 and M_3 on the drain with angles at them equal to 2π in flow schemes 1 and 3 (see Fig. 2). We fix the constants $t_6 = 0$, $t_2 = 1$, and t_0^m ($m = 1, 2, 3$) (the superscript m indicates the number of the flow scheme) assuming that $t_0^2 = -1$, $t_0^3 = (t_2 - t_3)/2$, and $t_0^1 = (t_4 - t_5)/2$.

The functions $d\omega^m/d\zeta$, $\omega^m : E \to D_m^*$ and $dZ^m/d\zeta$, Z^m: $E \to D$ are represented in the form

$$\frac{d\omega^m}{d\zeta} = K(\zeta - t_0^m)[(\zeta - t_6)(\zeta - t_5)]^{-1}(\zeta - t_4)^{-1/2} = \Pi_0^m(\zeta),$$

$$\frac{dZ^m}{d\zeta} = -\frac{\Pi(\zeta)}{\pi} \int\limits_{\Omega} \frac{\sigma|\Pi_0^m(t)|\,dt}{\Pi(t)(t-\zeta)}, \quad \Pi = (\zeta - t_2)(\zeta - t_3)(\zeta - t_6)^{\alpha-2}(\zeta - t_5)^{-1-\alpha},$$

where $\Omega = (-\infty, 0) \cup (t_5, t_4)$, $K = K(Q_0, Q_1)$ is a given constant [5, p. 233], $\sigma = \operatorname{sign}(dy/dt)$, and $t \in \Omega$. On the assumption that the upstream and downstream depths h_0^m and h_1^m are given, we calculate the discharges Q_0^m and Q_1^m: $Q_j^m = Kh_j^m \sin(\alpha\pi)\cos(\alpha\pi)$, $j = 0, 1$ [5, pp. 234–235]. We have $Q_0^m > Q_1^m$ ($m = 1$, 3) in schemes 1 and 3, and $Q_0^2 < Q_1^2$ in scheme 2 (see Fig. 3).

The constants t_3, t_4, and t_5 are calculated from the following system of equations:

$$b_1 = \int\limits_{t_4}^{\infty} \left|\frac{dZ^m}{dt}\right| dt, \quad \pi Q_j^m = |\Pi_0^m(\zeta)(\zeta - t_{6-j})|_{\zeta = t_{6-j}}, \quad j = 0, 1. \tag{11}$$

Here $b_1 = |Z(t_1) - Z(t_4)|$ is the drainage length (see Fig. 2) and discharges Q_j^m are given quantities.

Theorem 4. *Problems 1–3 of liquid filtration to the drain in the presence of an inclined water-confining stratum are uniquely solvable, and the solutions* (t_3, t_4, *and* t_5) *of system* (11) *corresponding to them satisfy the inequalities*

$$t_k - t_{k+1} \geqslant \varepsilon > 0, \qquad k = 2, 3, 4, 5. \tag{12}$$

Proof. Proof of the theorem statements, as previously, follows from the correctness of the *a priori* estimates (12). Let us write out two last equations of (11) in detail:

$$\pi Q_0^m = K|t_6 - t_0^m|(t_5 - t_6)^{-1}(t_4 - t_6)^{-1/2}, \quad \pi Q_1^m = K|t_5 - t_0^m|(t_5 - t_6)^{-1}(t_4 - t_5)^{-1/2}.$$

If $t_5 \to t_6 = 0$ or $t_4 \to t_5$, then $Q_j^m \to \infty$ $(j = 0, 1)$, which proves inequalities (12) for $k = 4$ and 5. To prove the remaining inequalities of (12), let us consider several cases, as in Sec. 3.

Assumptions: $t_2 - t_3 = r \to 0$ and $t_3 - t_4 \geqslant \varepsilon > 0$. Then, we have

$$b_1 \leqslant b = \int\limits_{t_3}^{t_2} \left|\frac{dZ^m}{dt}\right| dt = r^3 \int\limits_0^1 \Lambda(s, r)\, ds \to 0 \quad \text{at} \quad r \to 0.$$

Here the variables $t = sr + t_3$, $s \in [0, 1]$ are replaced in the integral, and it is taken into account that $\int\limits_0^1 \Lambda(s, r)\, ds \leqslant N_0 < \infty$.

Assumptions: $t_3 - t_4 = r \to 0$ and $t_2 - t_3 \geqslant \varepsilon > 0$. Then, we have

$$I \equiv \int\limits_{t_5}^{t_4} \frac{\sigma|\Pi_0(t)|\, dt}{\Pi(t)(t - \zeta)} \to \infty \quad \text{at} \quad r \to 0, \ s \in (-\infty, \infty),$$

since $|\Pi_0(t)|\,|\Pi^{-1}(t)| \leqslant K_1|t_3 - t|^{-3/2}$ at $t_4 = t_3$. Here

$$|z_* - z_2| = \int\limits_{t_*}^{t_2} \left|\frac{dZ^m}{dt}\right| dt \to \infty \quad \text{as} \quad r \to 0, \ t_* \in [t_3, t_2).$$

In particular, for the constant t_* corresponding to the point $z_* = Z(t_*) = b_1$ on the drain M_3M_2, it also follows that $z_* = b_1 \to \infty$.

Assumption: $t_2 - t_4 = r \to 0$.

In this case, also $I \to \infty$ and, thus, $z_* \to \infty$.

The obtained contradictions to the inequality $|\ln b_1| < \infty$ prove estimates (12) for $k = 2, 3$. The theorem is proved.

Remark 1. In the monograph [5, p. 231–239], only the drain length b_1 was considered as given, the constants $t_6 = 0$ and $t_5 = 1$ were fixed, and t_4 was derived from the first equation of (11). The presence of the points M_2 and M_3 with angles at them equal to 2π was not taken into consideration, and thus, the mapping $Z\colon E \to D$ was independent of the constants t_2 and t_3. Equation (11) with respect to $t_4 > 0$ was solved numerically in [5]. We were the first to solve problems 1–3 in the direct formulation.

Remark 2. Similar treatment is applied to the case where, in the vicinity of the infinite points M_5 and M_6, the branches of free boundaries emanating from of M_1 and M_4 can reach the impermeable top of the water-bearing stratum — segments of the straight lines $M_5M_5^*$ and $M_6M_6^*$ parallel to the confining bed M_6M_5. At the same time, the specified segments of the straight lines $M_5M_5^*$ and $M_6M_6^*$ and confining bed M_6M_5 can be substituted for the polygon P with the apices z_k $(k = \overline{0, n+1})$, where $z_s = \infty$ corresponds to the point M_6, and $z_m = \infty$ $(m > s)$ corresponds to the point M_5.

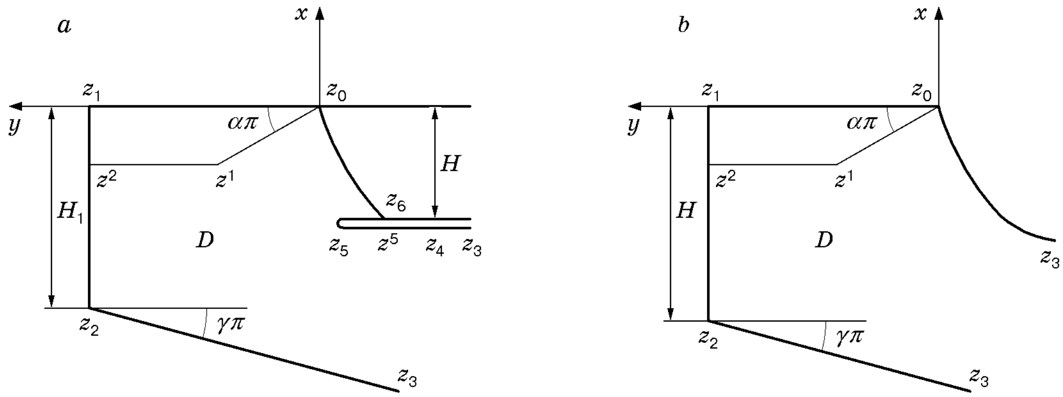

Fig. 4

7. FILTRATION LIQUID FLOW FROM A CHANNEL INTO AN INCLINED CONFINING BED

Similar problems were studied in the monograph [5, p. 147, 167]; schemes of the filtration-flow domain are shown in Fig. 4.

7.1. Liquid Filtration from a Rectilinear Channel into a Horizontal Water Intake above a Inclined Confining Bed (Fig. 4a)**.** In this problem, the channel bottom $P_1 = \{z : x = 0,\ 0 < y < y_1\}$ and drainage $P_5 = \{z\colon\ x = -H,\ y \in (y_4, y_5) \cup (y_6, y_5)\}$ are equipotentials, $\varphi = 0$ and $\varphi = H$, respectively. The axis of symmetry $P_2 = \{z\colon\ -H_1 = x_2 < x < 0,\ y = y_1 > 0\}$ and the confining bed $P_3 = \{z\colon\ -\infty < x < -H_1,$ $y - y_2 = (x + H_1)\cot(\gamma\pi)\}$ are a streamline $\psi = 0$; the free boundary L and the top of the water-bearing stratum $P_4 = \{z\colon\ x = -H,\ -\infty < y < y_4\}$ are also streamlines $\psi = Q$ and $\psi = Q_1 < Q$, Q and Q_1 are the sought liquid discharges.

In the plane of the complex potential $w = \varphi + i\psi$, to the domain D ($\partial D = L \cup P$, where $P = \bigcup_1^5 P_k$) there corresponds a half-band with a step D^* with the apices w_k and angles $\gamma_k\pi$ at them: $w_0 = iQ$, $w_1 = 0$, $w_2 = \varphi_2$, $w_3 = \infty$, $w_4 = H + i(Q - Q_1)$, $w_5 = H + i\psi_5$, $w_6 = H + iQ$; $\gamma_0 = \gamma_1 = \gamma_6 = 1/2$, $\gamma_2 = \gamma_5 = 1$, $\gamma_3 = 0$, and $\gamma_4 = 3/2$.

The derivatives $dw/d\zeta$ ($w\colon\ E \to D^*$) and $dz/d\zeta$ ($z\colon\ E \to D$) are represented in the form

$$\frac{dw}{d\zeta} = K\mathrm{e}^{i\beta\pi}\prod_{k=0}^{6}(\zeta - t_k)^{\gamma_k - 1} \equiv \Pi_0(\zeta), \qquad \frac{dz}{d\zeta} = \Pi(\zeta)M(\zeta),$$
$$\Pi = \prod_{k=0}^{6}(\zeta - t_k)^{\alpha_k - 1}, \qquad M = \frac{1}{\pi i}\int_{-1}^{1}\frac{|\Pi_0(t)|\,dt}{\Pi(t)(t - \zeta)}, \tag{13}$$

where $\alpha_0 = \alpha_6 = 1$, $\alpha_1 = 1/2$, $\alpha_2 = 1/2 + \gamma$, $\alpha_3 = -\gamma$, $\alpha_4 = 1 - \gamma$, and $\alpha_5 = 2$. The constants $K = 1$, $t_0 = 1$, $t_6 = -1$, and $t_k = 1 + k$ ($k = 4, 5$) are fixed, and the constants t_1, t_2, and t_3 and $t^5 \in (t_4, t_5)$ are found from the system of equations

$$l_k = \int_{t_{k-1}}^{t_k}\left|\frac{dz}{dt}\right|dt, \quad k = 1, 2; \qquad l = \int_{t_4}^{t^5}\left|\frac{dz}{dt}\right|dt, \quad H = \int_{-1}^{1}|\Pi_0(t)|\,dt. \tag{14}$$

Here the quantities H, l_1, l_2, and l are given: $H = |w_6 - w_0|$, $l_1 = |z_1 - z_0| = y_1$, $l_2 = |z_2 - z_1| = H_1$, the length of the drainage slot is $l = |z^5 - z_4|$; $y^5 \in (y_4, y_5)$ ($z^5 = z_6$). Note that the ordinates y_k of the points z_k ($k = 4, 5, 6$) are not fixed.

A priori estimates of the solution of system (14) $0 < \varepsilon \leqslant t_{k+1} - t_k$ ($k = \overline{0,3}$) and $0 < \varepsilon \leqslant |t^5 - t_k|$ ($k = 3, 4, 5$) are established in the same way as in Sec. 3. From these estimates and the results of [1], there follows the unique solvability of the original problem.

7.2. Liquid Filtration from a Rectilinear Channel into an Inclined Confining Bed. Analogous problems are studied in [6, pp. 308, 318, and 331] for the case of a horizontal water-confining stratum and in [5, p. 167] for the case of its absence.

The channel bottom $P_1 = \{z\colon\ x = 0,\ 0 < y < y_1\}$ is an equipotential $\varphi = 0$, the axis of symmetry $P_2 = \{z\colon -H < x < 0,\ y = y_1\}$ and the confining bed $P_3 = \{z\colon -\infty < x < -H,\ y - y_2 = (x + H)\cot(\gamma\pi)\}$ are a streamline $\psi = 0$; on the free boundary L, we have $\psi = Q$, which is the sought liquid discharge.

To the filtration domain D ($\partial D = P \cup L$, where $P = \bigcup_1^3 P_k$) in the plain $w = \varphi + i\psi$, there corresponds a half-band D^* with the apices w_k and angles $\gamma_k\pi$ at them: $w_0 = iQ$, $w_1 = 0$, $w_2 = H$, $w_3 = \infty$, $\gamma_0 = \gamma_1 = 1/2$, $\gamma_2 = 1$, $\gamma_3 = 0$.

The derivatives of the conformal mappings $w\colon E \to D^*$ and $z\colon E \to D$ are represented in the form (13), where the products Π are taken Π_0 in the range from 0 to 3; and $\alpha_0 = 1$, $\alpha_1 = 1/2$, $\alpha_2 = 1/2 + \gamma$, and $\alpha_3 = 1 - \gamma$.

The constants $K = 1$, $t_0 = 1$, and $t_3 = -1$ are fixed, and t_1 and t_2 are found from the following system of equations of the form (14):

$$l_k = |z_k - z_{k-1}| = \int\limits_{t_{k-1}}^{t_k} \left|\frac{dz}{dt}\right| dt, \quad k = 1, 2.$$

A priori estimates $0 < \varepsilon \leqslant t_{k+1} - t_k \leqslant \varepsilon^{-1}$ $(k = 0, 1)$ and unique solvability are proved similarly to Secs. 3 and 6.

Note, in the vicinity $|\zeta - t_3| \leqslant 1$ of the point t_3, there holds the inequality $|M(\zeta)(\zeta - t_3)^{1-\gamma}| \leqslant N < \infty$, hence, $|dz(\zeta - t_3)/d\zeta| \leqslant N_0 < \infty$, which corresponds to the zero angle ∂D at the point $z_3 = \infty$ (L and the confining bed P_3 are parallel for $z \to \infty$).

7.3. Channel Bottom and Confining Beds As Arbitrary Polygonal Boundaries. Figure 4 shows the case of a trapezoidal channel bottom studied in [5, p. 167–181] in the absence of a confining bed. All the calculations in Secs. 3 and 6 hold true for this case, too. At the same time, the form of the derivative $dw/d\zeta$ is unchanged, and the product $\Pi(\zeta)$ for the domain D shown in Fig. 4a is represented as

$$\Pi(\zeta) = \prod_{k=2}^{6} (\zeta - t_k)^{\alpha_k - 1}\Pi_*(\zeta), \quad \Pi_* = \left(\frac{\zeta - t^1}{\zeta - t_0}\right)^{\alpha} (\zeta - t^2)^{-1/2} \quad (k = \overline{2, 6})$$

(α_k are the same as in Sec. 7.1); for the domain D in Fig. 4b, it has the form

$$\Pi(\zeta) = \prod_{k=2}^{3} (\zeta - t_k)^{\alpha_k - 1}\Pi_*(\zeta)$$

(α_2 and α_3 are the same as in Sec. 7.2). The constants t^k $(k = 1, 2)$ are the preimages of the points $z^k = z(t^k)$.

Moreover, the results obtained in Secs. 3–7 are also valid for the case where the channel bottom and confining beds have the form of polygons with a finite number of apices.

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